Solutions to Math 208 Midterm I

October 20, 2023

(1) **Consider following system of equations to answer the questions below.**

$$
\begin{aligned}
x_1 + 8x_2 + 3x_3 + x_4 &= 12 \\
x_2 + x_3 + x_4 &= 1 \\
x_1 + 2x_2 + 3x_3 &= 1 \\
2x_4 &= 2
\end{aligned}
$$

(a) (5 points) Write down the augmented matrix for this system. a)

Augmented matrix for Test Code: 4050.

$$
A = \left[\begin{array}{cccc|c}
1 & 8 & 3 & 1 & 12 \\
0 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 0 & 1 \\
0 & 0 & 0 & 2 & 2
\end{array}\right]
$$

(b) (15 points) Solve this system by computing the REDUCED row echelon form for the augmented matrix by Gauss-Jordan elimination and label each step. (Hint: think about what steps to do first to make this process more efficient.)

b) Ok, let's roll up our sleeves...

$$A \xrightarrow[R_4/2\to R_4]{1} \left[\begin{array}{cccc|c} 1 & 8 & 3 & 1 & 12 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2-R_4\to R_2]{R_1-R_4\to R_1} \left[\begin{array}{cccc|c} 1 & 8 & 3 & 0 & 11 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1-8R_2\to R_1]{} \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 11 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_3-R_1\to R_3]{} \left[\begin{array}{cccc|r} 1 & 0 & -5 & 0 & 11 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 8 & 0 & -10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2\to R_1]{} \left[\begin{array}{cccc|r} 1 & 0 & -5 & 0 & 11 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_3/6\to R_3]{} \left[\begin{array}{cccc|r} 1 & 0 & -5 & 0 & 11 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-R_3\to R_2]{R_1+5R_3\to R_1} \left[\begin{array}{cccc|r} 1 & 0 & 0 & 0 & 8/3 \\ 0 & 1 & 0 & 0 & 5/3 \\ 0 & 0 & 1 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Phew! Just 7 steps. So there is a unique solution the system of 4 equations and 4 unknowns given above. It is $x_1 = 8/3$, $x_2 = 5/3$, $x_3 = -5/3$, $x_4 = 1$.

Check it:

$$\begin{aligned} 8/3 + 8 * 5/3 - 3 * 5/3 + 1 =& 12 \\ 5/3 - 5/3 + 1 =& 1 \\ 8/3 + 2 * 5/3 - 3 * 5/3 =& 1 \\ 2 * 1 =& 2 \end{aligned}$$

Good! That was how it was supposed to work.

(2) **The Math Minded Knitters Consortium (MMKC) are making hats this winter. They use three types of yarn with different compositions of materials determined by the table below. Each type of yarn is the same weight.**

Materials	Brand A	Brand B	Brand C
merino wool	70%	80%	90%
possum	10%	10%	10%
silk	10%	0%	0%
recycled plastic	10%	10%	0%

(a) (5 points) Write down the matrix equation that would determine the total number of grams of wool, possum, silk, and recycled plastic that are in a grams of Brand A yarn, b grams of Brand B yarn, and c grams of Brand C yarn. Explain the notation in the right side in your matrix equation for the members of MMKC.

The equation will be of the form $A\mathbf{x} = \mathbf{b}$, but we will use variables related to this problem. Here we have

$$\begin{bmatrix} .7 & .8 & .9 \\ .1 & .1 & .1 \\ .1 & 0 & 0 \\ .1 & .1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} m \\ p \\ s \\ r \end{bmatrix},$$

where m is the total number of grams of merino wool, p be the total number of grams of possum wool, s be the total number of grams of silk, and r be the total number of grams of recycled plastic in a composition with a grams of Brand A yarn, b grams of Brand B yarn, and c grams of Brand C yarn.

(b) (5 points) Use your matrix equation to determine how many grams of each material are in a hat with 50grams of Brand A yarn, 20 grams of Brand B yarn, and 30 grams of Brand C yarn? Test code: 8745's numbers.

$$\begin{bmatrix} .7 & .8 & .9 \\ .1 & .1 & .1 \\ .1 & 0 & 0 \\ .1 & .1 & 0 \end{bmatrix} \begin{bmatrix} 50 \\ 20 \\ 30 \end{bmatrix} = 5 \begin{bmatrix} 7 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 8 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 9 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 10 \\ 5 \\ 7 \end{bmatrix}$$

Check: $78 + 10 + 5 + 7 = 100$. The grams of each material should sum to 100 in this case, since we are mixing 100 grams of yarn.

(c) (10 points) Can MMKC make 100 gram hats with 83 grams of wool, 10 grams of possum, 3 grams of silk, and 4 grams of recycled plastic? If so, how much of each brand of yarn should they use? If not, explain why not.

Sure! We could solve the corresponding matrix equation. Or, we can look carefully at the data. To get 3 grams of silk, we will need 30 grams of Brand A yarn. Then to also get 4 grams of recycled plastic, we will need an additional 10 grams of Brand B yarn. Luckily, it works out just right if we take the remaining 60 grams to be all Brand C.

$$30\begin{bmatrix} .7 \\ .1 \\ .1 \\ .1 \end{bmatrix} + 10\begin{bmatrix} .8 \\ .1 \\ 0 \\ .1 \end{bmatrix} + 60\begin{bmatrix} .9 \\ .1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 83 \\ 10 \\ 3 \\ 4 \end{bmatrix}$$

(3) **Consider the matrices A and B in each question below.**

$$A = \begin{bmatrix} 1 & -1 & 2 & 8 & 1 \\ 0 & 1 & -3 & 8 & 0 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 2 & 1 & 1 \\ 0 & 7 & -3 & 10 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) (5 points) Which subsets of columns of A are linearly dependent? Explain your answer using an Echelon Test Theorem.

Answer: $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 4, 5\}$.

Explanation: Circle the pivots in A. They are in columns $\{1, 2, 3, 5\}$, so these columns are independent, and all subsets of this set index independent columns. To get a dependent set we add one more column, but 4 is the only additional column. So $\{1, 2, 3, 4, 5\}$ indexed a set of dependent columns. Note that among columns $\{1, 2, 3, 4\}$ there are fewer pivots than columns and this submatrix is still in echelon form. Therefore, the Echelon Test still applies here and proves $\{1, 2, 3, 4\}$ indexes a dependent set of columns.

(b) (5 points) Which subsets of columns of B span all of $\mathbb{R}^4$? Explain your answer using an Echelon Test Theorem.

Answer: $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$, and $\{1, 2, 3, 4, 5\}$.

Explanation: Circle the pivots in B. They are in columns $\{1, 2, 4, 5\}$, so these columns span all of $\mathbb{R}^4$ since there is a pivot in every row by the Echelon Test for Spanning Vectors. Every set containing $\{1, 2, 4, 5\}$ also spans, so $\{1, 2, 3, 4, 5\}$ indexes a spanning set also. Finally, note that if we consider the submatrix on columns $\{1,3,4,5\}$, it is also in echelon form and would have a pivot in every row if we recomputed the pivots without column 2.

No to prove this is all of them, note every subset of less than 4 columns cannot span $\mathbb{R}^4$. It remains to check $\{1,2,3,4\}$ which has no nonzero entry in the bottom row, and $\{1,2,3,5\}$ which has no nonzero entry in the 3rd row. Hence, neither of these can span $\mathbb{R}^4$.

(c) (5 points) Compute $C = A + 3B$.

$$C = A+3B = \begin{bmatrix} 1 & -1 & 2 & 8 & 1 \\ 0 & 1 & -3 & 8 & 0 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 6 & 6 & 3 & 3 \\ 0 & 21 & -9 & 30 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 5 & 8 & 11 & 4 \\ 0 & 22 & -12 & 38 & 0 \\ 0 & 0 & 1 & 20 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

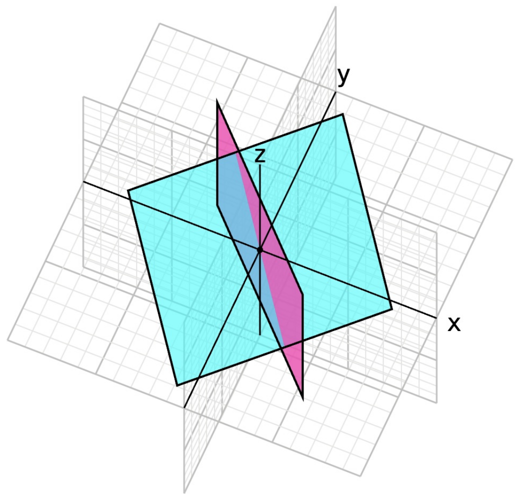
(d) (5 points) Which subsets of columns of C are linearly dependent and which ones span $\mathbb{R}^4$? Explain your answer.

Answer: The columns indexed by $\{1,2,3,4\}$ and $\{1,2,3,4,5\}$ are linearly dependent in C. The columns indexed by $\{1,2,3,5\}$, $\{1,2,4,5\}$, $\{1,3,4,5\}$, $\{2,3,4,5\}$, and $\{1,2,3,4,5\}$.

Explanation: Similar reasoning to the work above. Note $\{1,2,3,4\}$ is not a spanning set of columns in C. All other sets of size 4 or 5 will span.

(4) **Let S be the subspace of $\mathbb{R}^3$ determined by intersecting the plane given by $x - y + z = 0$ and the plane given by $x + y = 0$.**

(a) (5 points) What familiar type of geometrical object is S?
These two planes intersect in a line, which is the most common case. The other options are that they are the same plane, or they are parallel and don't meet at all.



(b) (5 points) What column vectors are in S?

We need to solve $x - y + z = 0$ and by $x + y = 0$. Using the augmented matrix and Gauss elimination is helpful to see z is a free variable.

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

Set $z = t$, then the bottom row says $-2y + z = 0$ so $y = t/2$. The top row says $x + y = 0$, so $x = -t/2$. Therefore, as column vectors the solution set

$$S = \left\{ t \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \ : \ t \in \mathbb{R} \right\}.$$

Equivalently, S is all scalar multiples of $[1, 1, 2]^t$.

(c) (10 points) Give the equation of a plane that does not intersect S. Explain why the intersection is empty.
The plane $x + y = 1$ is parallel to $x + y = 0$, so it also does not intersect S.

(5) **Consider the vector equation**

$$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

(a) (10 points) What is the general solution?

Answer: $w = -34 - s, x = -6 - s, y = s, z = 6$ where s can be any real number.

Explanation: This system is already in row echelon form, so it's just a matter of translating it back into equations and using back-substitution. In this form, we can deduce w, x, z are leading variables and y is a free variable. Let $y = s$, where s can take on the value of any real number. The bottom row of the vector equation tells us that $z = 6$. The second row tells us $x+y+z = 0$, and since $y = s$ and $z = 6$ we deduce $x = -6 - s$. From the top row we know $w + x + 2y + 7z = 2$. Plugging in the values found for x, y, z and simplifying, we get $w = -34 - s$.

(b) (10 points) Give a specific solution that involves a nonzero combination of all 4 vectors.

Answer: Take $s = 1$ above. Then $w = -35, x = -7, y = 1, z = 6$ is a specific solution. And, then check this really is a solution by plugging these values into the original vector equation.

$$-35 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 7 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -35 - 7 + 2 + 42 \\ 0 - 7 + 1 + 6 \\ 0 + 0 + 0 + 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

Bonus: (2pt) What have you found easiest and hardest so far in Math 208? Do you wish the pace to go FASTER, SLOWER, or ABOUT THE SAME?

(Or if you don't want to answer those two questions, draw a picture involving vectors here.)

Great feedback! Thanks!

- Most common: about the same ($\sim 70\%$), then slower ($\sim 20\%$), and some people said faster ($\sim 10\%$).
- Easiest: adding matrices, Gauss elimination, span, multiplying matrices, webassign problems.
- Hardest: Conceptual problems, linear independence, span, Gauss elimination, multiplying matrices, waking up in time for class, avoiding covid.
- Question: How do you get the equation of a plane in $\mathbb{R}^3$?

Quote: The hardest part is that there aren't many straightforward problems/answers like usually with math where you can plug and chug. This course requires a lot more thinking and problem solving.